Control Systems : Set 11 : Statespace (2) - Solutions

Prob 1 | Consider a system with state matrices

$$A = \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 3 \end{bmatrix}$$

a) Use feedback of the form $u(t) = -Kx(t) + \bar{N}r(t)$, where $\bar{N}$ is a nonzero scalar, to move the poles to $-3 \pm 3j$

> The reference has no impact on the pole locations. The poles are given by
>
> $$\det(sI - A + BK) = (s + 2 + K_1)(s + 3 + K_2) - (K_2 - 1)K_1 = s^2 + 6s + 18$$
>
> Equating coefficients and solving gives $K = \begin{bmatrix} 5 & -4 \end{bmatrix}$

b) Choose $\bar{N}$ so that if r is a constant, the system has zero steady-state error, that is $y(\infty) = r$

> We choose $\bar{N}$ so that the DC gain between r and y is unity. DC occurs when the derivative is zero
>
> $$\dot{x} = 0 = (A - BK)x + B\bar{N}r \qquad \rightarrow \qquad x = -(A - BK)^{-1}B\bar{N}r$$
> $$y = Cx = -C(A - BK)^{-1}B\bar{N}r$$
> $$= \frac{5}{9}\bar{N}r$$
>
> So to have a DC gain of one, we need
>
> $$C(A - BK)^{-1}B\bar{N} = \frac{5}{9}\bar{N} = 1 \qquad \rightarrow \qquad \bar{N} = \frac{9}{5}$$

c) The system steady-state error performance can be made robust by augmenting the system with an integrator and using unity feedback, that is, by setting $\dot{x}_I = r - y$, where x_I is the state of the integrator. To see this, first use state feedback of the form $u = -Kx - K_I x_I$ so that the poles of the augmented system are at -3, $-2 \pm j\sqrt{3}$

> Adding an integrator to the system enhances the system dynamics to
>
> $$\begin{bmatrix} \dot{x} \\ \dot{x}_I \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} \begin{bmatrix} x \\ x_I \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$
> $$y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ x_I \end{bmatrix}$$
>
> We design a state-feedback controller in the usual fashion for this new, larger system

via the place function in Matlab.

$$K = \begin{bmatrix} 0.3 & 1.7 & -2.1 \end{bmatrix}$$

Prob 2 | For the system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

design a state feedback controller that satisfies the following specifications:

- Closed-loop poles have a damping coefficient $\zeta = 0.707$
- Step-response peak time is under 3.14sec

The peak time is related to the damped frequency ω_d

$$T_p = \frac{\pi}{\omega_d}$$

To achieve a peak time of 3.14, we need a damped frequency of 1. Solving for ω_n gives

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = 1 \qquad \rightarrow \qquad \omega_n = 1.414$$

Our desired characteristic equation is therefore

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 2s + 2$$

Our state feedback controller takes the form $u = -Kx$, and the resulting closed-loop characteristic equation is

$$\begin{aligned} \det(sI - (A - BK)) &= \left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ K_1 & K_2 \end{bmatrix} \right| \\ &= \left| \begin{bmatrix} s & -1 \\ 6 + K_1 & s + 5 + K_2 \end{bmatrix} \right| \\ &= s(s + 5 + K_2) + 6 + K_1 \\ &= s^2 + (5 + K_2)s + 6 + K_1 \end{aligned}$$

We equate the coefficients to get

$$\begin{aligned} 5 + K_2 &= 2 \qquad \rightarrow \qquad K_2 = -3 \\ 6 + K_1 &= 2 \qquad \rightarrow \qquad K_1 = -4 \end{aligned}$$

Prob 3 | Consider the following system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -10 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

a) Design a state feedback controller so that the closed-loop step response has an overshoot of less than 25% and a 1% settling time under 0.115sec

The given criteria translate into pole locations:

$$\zeta \geq \frac{-\log(0.25)}{\sqrt{\log(0.25)^2 + \pi^2}} = 0.4$$
$$T_s = \frac{-\log(0.01)}{\zeta\omega_n} = 0.115 \qquad \rightarrow \qquad \omega_n = 99$$

Which gives a desired characteristic equation of

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 80s + 9'839$$

The closed-loop system will be

$$A - BK = \begin{bmatrix} 0 & 1 \\ 0 & -10 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ K_1 & K_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -K_1 & -10 - K_2 \end{bmatrix}$$

which has the characteristic equation

$$s^2 + (10 + K_2)s + K_1$$

Equating the coefficients to our desired equation gives

$$K = \begin{bmatrix} 9'839 & 70 \end{bmatrix}$$

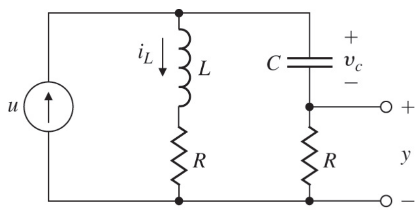
b) Use the step command in Matlab to verify that your design meets the specifications. If it does not, modify your feedback gains accordingly.

The following matlab commands solves the problem

```
% System equations
A = [0 1 ; 0 -10];
B = [0;1];
C = [1 0];
D = 0;

% Specifications
Mp = 0.25;
Ts = 0.115;
SettlingTimeThreshold = 0.01;

% Compute target poles
```

```
zeta = -log(Mp)/sqrt(log(Mp)^2 + pi^2);
wn = -log(SettlingTimeThreshold)/(zeta*Ts);
p = roots([1 2*zeta*wn wn^2]);

% Place the poles
K = place(A, B, p)

K =

         9839        70.09


% Simulate the closed-loop system
cl_sys = ss(A-B*K, B, C, D);
stepinfo(cl_sys, 'settlingtimethreshold', SettlingTimeThreshold )

ans =

  struct with fields:

        RiseTime: 0.014832
    SettlingTime: 0.11395
     SettlingMin: 9.5286e-05
     SettlingMax: 0.00012704
       Overshoot: 24.998
      Undershoot: 0
            Peak: 0.00012704
        PeakTime: 0.0345
```

Prob 4 | Consider the electric circuit shown in the figure below, that you designed a controller for in the fifth exercise



a) What condition(s) on R, L and C will guarantee that the system is observable?

Recall the state-space representation derived in the previous exercise set

$$\begin{pmatrix} \dot{i}_L \\ \dot{v}_c \end{pmatrix} = \begin{bmatrix} -\frac{R}{L} & \frac{1}{L} \\ -\frac{1}{C} & 0 \end{bmatrix} \begin{pmatrix} i_L \\ v_c \end{pmatrix} + \begin{pmatrix} \frac{R}{L} \\ \frac{1}{C} \end{pmatrix} u$$

$$y = \begin{bmatrix} -R & 0 \end{bmatrix} \begin{pmatrix} i_L \\ v_c \end{pmatrix} + Ru$$

The condition for the system to be unobservable is

$$
\begin{aligned}
\det(\mathcal{O}) &= 0 \\
&= \left| \begin{bmatrix} C \\ CA \end{bmatrix} \right| = \left| \begin{bmatrix} -R & 0 \\ \frac{R^2}{L} & -\frac{R}{L} \end{bmatrix} \right| \\
&= \frac{R^2}{L}
\end{aligned}
$$

As the determinant is non-zero for all positive R and L, the system will be observable for all parameters.

Prob 5 | Consider the system

$$A = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

and assume that you are using feedback of the form $u = -Kx + r$ where r is a reference input signal

a) Show that (A, C) is observable

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

is nonsingular. Therefore (A, C) is observable.

b) Show that there exists a K such that $(A - BK, C)$ is unobservable

The observability matrix of the closed-loop system is

$$\mathcal{O} = \begin{bmatrix} C \\ C(A - BK) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -K_1 & 1 - K_2 \end{bmatrix}$$

The system is unobservable when the observability matrix looses rank.

$$\det \mathcal{O} = 0 = 2K_1 - K_2 + 1$$

c) Compute a K of the form $K = \begin{bmatrix} 1 & K_2 \end{bmatrix}$ that will make the system unobservable as in part (b), that is, find K_2 so that the closed-loop system is not observable

Solving the expression derived in the previous question gives

$$0 = 2K_1 - K_2 + 1 \qquad \rightarrow \qquad K_2 = 3$$

d) Compare the open-loop transfer function with the transfer function of the closed-loop system of part (c). What is the unobservability due to?

$$\begin{aligned} G_{ol}(s) = C(sI - A)^{-1}B &= \frac{s+2}{s^2 + 2s - 1} = \frac{s+2}{(s + 2.414)(s - 0.4142)} \\ G_{cl}(s) = C(sI - A + BK)^{-1}B &= \frac{s+2}{s^2 + 3s + 2} = \frac{s+2}{(s+2)(s+1)} \end{aligned}$$

We see that the closed-loop system is unobservable due to pole-zero cancellation, meaning that we don't see the impact of these states at the output of the system.